Schur duality

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Outline

1. Basics of representation theory
2. Schur duality
3. Applications

Basics of representation theory

Representation

Definition

A representation $(\phi, \mathbb{C}^n)$ over the vector space $\mathbb{C}^n$ of a group G is a homomorphism $\phi : G \to \mathrm{GL}(n, \mathbb{C})$.

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- Homomorphism: $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ for all $g_1, g_2 \in G$
- $\mathrm{GL}(n, \mathbb{C})$: $n \times n$ invertible complex matrices

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- $(\phi_{\det}, \mathbb{C})$ given by $\phi_{\det}(U) = \det(U)$

Direct sum and tensor product

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Let (ϕ_1, V_1) and (ϕ_2, V_2) be representations of G. Then representations $(\phi_1 \oplus \phi_2, V_1 \oplus V_2)$ and $(\phi_1 \otimes \phi_2, V_1 \otimes V_2)$ of G are their direct sum and tensor product, respectively.

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Let $(\phi_1, \mathbb{C}^2)$, $(\phi_2, \mathbb{C})$ be representations of $\mathcal{U}(2)$ such that

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Then $(\phi_1 \oplus \phi_2, \mathbb{C}^3)$ is their direct sum and $(\phi_1 \otimes \phi_2, \mathbb{C}^2)$ is their tensor product.

$$(\phi_1 \oplus \phi_2)(U) = U \oplus 1 = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \qquad (\phi_1 \otimes \phi_2)(U) = U \otimes 1 = U$$

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Theorem

Every representation (ϕ, V) of G is isomorphic to a direct sum of irreducible representations of G:

$$\phi(g) \cong \bigoplus_{\lambda \in \hat{G}} \lambda(g) \otimes I_{n_\lambda}$$

Schur duality

Representations of $\mathcal{U}(d)$ and S_n

Consider representations

- $\left(\mathbf{Q}, \left(\mathbb{C}^d\right)^{\otimes n}\right)$ of $\mathcal{U}(d)$, where

$$\mathbf{Q}(U) \left| i_1 i_2 \ldots i_n \right\rangle = U \left| i_1 \right\rangle U \left| i_2 \right\rangle \ldots U \left| i_n \right\rangle$$

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Theorem. (Schur duality)

There exist a basis (Schur basis) in which representation $\left(\mathbf{QP}, \left(\mathbb{C}^d\right)^{\otimes n}\right)$ of $\mathcal{U}(d) \times S_n$ decomposes into irreducible representations $\mathbf{q}_\lambda$ and $\mathbf{p}_\lambda$ of $\mathcal{U}(d)$ and S_n respectively:

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Definition

Schur transform U_{sch} is unitary transformation implementing the base change from standard basis to Schur basis:

$$U_{\mathrm{sch}} = \sum_i |\mathrm{sch}_i\rangle \langle i|$$

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$$= \begin{pmatrix} \det(U)\,\mathrm{sgn}(\pi) & 0 \\ 0 & \mathbf{q}_{3\,\dim}(U) \end{pmatrix} \begin{matrix} |01\rangle - |10\rangle \\ |00\rangle\,,\ |11\rangle\,,\ |01\rangle + |10\rangle \end{matrix}$$

Applications

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- Universal distortion-free entanglement concentration using only local operations.
  1. Each party applies Schur transform
  2. Measure $\lambda \in \mathrm{Par}(n, d)$. Discard $\mathcal{Q}_\lambda$, retaining $\mathcal{P}_\lambda$.
  3. A and B share maximally entangled state of dimension $\dim(\mathcal{P}_\lambda)$
- Encoding/decoding into decoherence free subspaces

Thank you!

Outline of proof for Schur duality

Every representation can be expressed as a direct sum of irreps:

$$\mathbf{P}(\pi) \stackrel{S_n}{\cong} \bigoplus_{\lambda \in \hat{S_n}} \mathbf{p}_\lambda(\pi) \otimes I_{n_\lambda} \qquad \mathbf{Q}(\mathbf{U}) \stackrel{U_d}{\cong} \bigoplus_{\lambda \in \hat{U}_d} \mathbf{q}_\lambda(U) \otimes I_{n_\lambda}$$

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Since $\mathbf{P}(\pi)$ and $\mathbf{Q}(\mathbf{U})$ commute, via Schur's lemma we get

$$\mathbf{Q}(\mathbf{U})\mathbf{P}(\pi) \overset{U_d \times S_n}{\cong} \bigoplus_{\alpha} \bigoplus_{\beta} \mathbf{q}_\alpha(U) \otimes \mathbf{p}_\beta(\pi) \otimes I_{m_{\alpha,\beta}}$$

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Finally, it can be shown that the range of λ in previous formula corresponds to $\mathrm{Par}(n, d)$:

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